

NEMATIC STRUCTURE OF SPACE-TIME AND ITS TOPOLOGICAL DEFECTS IN 5D KALUZA-KLEIN THEORY

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Abstract

We show, that classical Kaluza-Klein theory possesses hidden nematic dynamics. It appears as a consequence of $1 + 4$ -decomposition procedure, involving 4D observers 1-form λ . After extracting of boundary terms the, so called, "effective matter" part of 5D geometrical action becomes proportional to square of anholonomicity 3-form $\lambda \wedge d\lambda$. It can be interpreted as twist nematic elastic energy, responsible for elastic reaction of 5D space-time on presence of anholonomic 4D submanifold, defined by λ . We derive both 5D covariant and $1 + 4$ forms of 5D nematic equilibrium equations, consider simple examples and discuss some 4D physical aspects of generic 5D nematic topological defects.

KEY WORDS: Kaluza-Klein theory, nematic structure, anholonomic manifold

1 Introduction

Up to a present time some fundamental concepts of continuum media mechanics have revealed their relevancy for more deep understanding of space-time physics [1]-[12]. As well as being evidence of interrelations and unity of such, at first glance, remoted physical topics, this fact also suggests that space-time and matter have unified geometro-physical base, providing both physical interpreting of some subtle geometrical properties of space-time and geometrical background for fundamental properties of matter.

In present paper we turn our attention to a classical Kaluza-Klein theory (KKT)[13, 14]. It attracts many theorists today due to profound insight of its central paradigm — extradimensions and its physical manifestations — on the one hand, and due to development of the theory within more contemporary framework on the other [15, 16, 17, 18, 19]. The more general (than original Kaluza-Klein) formulations and interpretations of the theory have allowed to establish unified geometrical background for a wide class of phenomena. Particularly, problem of fifth force, nature and origin of some fundamental classical notions (masses, charges), actual cosmological and astrophysical problems, some important aspects of quantum mechanics and elementary particle physics, especially, unification interactions problem — all this can be successfully "translated" on the generalized KKT language.

Let's remind that commonly used method of extracting 4D observable quantities from 5D world is $1+4$ -splitting procedure [20, 17]. 4D results of this procedure crucially depend on particular choice of 1-form λ , providing local splitting of 5D riemannian manifold on 4D space-time sections

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and extradirections. This brings up the question: what field λ is realized for a some 5D manifold with forgiven riemannian metrics G ? In previous works some authors have considered the freedom of choosing λ as an effective instrument for obtaining physically different 4D worlds from the same 5D manifold (the so called "generational procedure" [20, 21]). Other ones have restricted the freedom using some additional considerations (sometimes rather artificial).

In present paper we propose natural framework for answering to this question. It is based on variational procedure, applied to 4D part of 5D action. In brief, let 5D geometrical action¹

$${}^{(5)}\mathcal{A} = -\frac{1}{2\kappa_5} \int_{\mathcal{M}} {}^{(5)}R, \quad (1)$$

(where κ_5 , ${}^{(5)}R$ — 5D Einstein constant and scalar curvature respectively) is decomposed as follows:

$${}^{(5)}\mathcal{A} = {}^{(4)}\mathcal{A}[\lambda] + {}^{(4)}\mathcal{A}_m[\lambda]. \quad (2)$$

Here ${}^{(4)}\mathcal{A}[\lambda]$ — action for 4D gravity and ${}^{(4)}\mathcal{A}_m[\lambda]$ — action for 4D effective matter. If 5D metrics G , minimizing ${}^{(5)}\mathcal{A}$, is fixed, then ${}^{(5)}\mathcal{A}$ does not depend on λ , while the both terms in the righthand part of (2) depend (this is reflected in notations). We can assume, that *true λ is extremal for any of this two terms* (if one maximal, then other minimal and vice versa). For the definiteness we'll take $\delta {}^{(4)}\mathcal{A}_m[\lambda] = 0$ as equations, determining λ . At this point we'll reveal remarkable analogy of the problem to similar 3D problem for equilibrium deformations of nematic liquid crystals in continuum media physics [28]. We'll see that unit 4D observers field j_λ , dual to λ , plays role of director and endows 5D space-time nematic structure.

The plan of the paper is as follows.

In Sec. 2 we remind some basic ideas, relations and expressions of 5D KKT in frame of 1 + 4-splitting formalism. The aim of Sec.2 is expression (14) for lagrangian (up to the constant $-1/2\kappa_5$) for action ${}^{(4)}\mathcal{A}_m$ in (2).

Sec. 3 is devoted to nematic structure of 5D space-time, inspired by action ${}^{(4)}\mathcal{A}_m$ for effective matter. We remind general theoretical assumptions of nematic liquid crystal physics. Then we rewrite geometrical 4D lagrangian in the form, which clears analogy with nematic crystals and compare action ${}^{(4)}\mathcal{A}_m$ with elastic nematic free energy.

In Sec. 4 we derive equilibrium equations from action ${}^{(4)}\mathcal{A}_m$ together with boundary conditions.

In Sec.5 1 + 4-form of 5D nematic equilibrium equations is presented.

Small Sec. 6 touches some particular solutions, which can be observed in earlier works.

In Sec. 7 some examples of 5D nematic structures, satisfying equilibrium equations, are performed.

Conclusion contains general discussion.

In mathematical notations we follow mainly to [22, 23]. Particularly, we use the following notations and abbreviations:

ι_X and j_ω — 1-form and vector field dual to vector field X and 1-form ω respectively. For any vector field Y we have: $\iota_X(Y) \equiv \langle X, Y \rangle$, $\langle j_\omega, Y \rangle \equiv \omega(Y)$, where \langle , \rangle — riemannian metric;

$T_s^r(\mathcal{M})$ — r -contravariant and s -covariant tensor fields over \mathcal{M} ;

$T(\mathcal{M}) = \bigoplus_{r,s} T_s^r(\mathcal{M})$ — tensor algebra over \mathcal{M} ;

For any $T \in T_2^0(\mathcal{M})$ we define jT and Tj by the formulae:

$$(jT)(\omega, X) = T(j_\omega, X); \quad (Tj)(X, \omega) = T(X, j_\omega).$$

¹We assume $c = 1$ and supply all 4D and 5D quantities, denoted by the same letters with indexes (4) or (5) when it is necessary.

Coordinate form: $(jT)_\beta^\alpha = G^{\alpha\gamma}T_{\gamma\beta}$, $(Tj)_\beta^\alpha = G^{\alpha\gamma}T_{\beta\gamma}$ shows, that j can be viewed as coordinate free notation of tensor indexes raising. Lowering is defined similarly by means of \flat ;

\hat{S} and \hat{A} — symmetrization and antisymmetrization operators, acting in spaces $T_0^n(\mathcal{M})$ and $T_n^0\mathcal{M}$ for every n ; for example, in case $T \in T_2^0(\mathcal{M})$:

$$(\hat{S}T)(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X)); \quad (\hat{A}T)(X, Y) = \frac{1}{2}(T(X, Y) - T(Y, X));$$

$\nabla : T_s^r(\mathcal{M}) \rightarrow T_{s+1}^r(\mathcal{M})$ covariant (with respect to some fixed riemannian metrics G) derivative;
 $\text{Grad} \equiv j\nabla$; $\text{Div } X = \text{Tr}(\nabla X)$; $\text{Div } \omega = \text{Tr}(\text{Grad } \omega)$ — some useful differential operations, connected with ∇ . Here X and ω — arbitrary vector field and 1-form.

$\pi_X : \Lambda^p(\mathcal{M}) \rightarrow \Lambda^{p-1}(\mathcal{M})$ — lowering degree operator, acting on space of external forms of degree p by the rule:

$$(\pi_X \omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1});$$

$d\text{vol}_5 \equiv \sqrt{|G|}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^5$ — standard volume form on 5D riemannian manifold \mathcal{M} with metric G .

2 Essentials of 1 + 4—approach to 5D KKT

Present section is brief resemblance of some general ideas and relations of 1 + 4-splitting technic (monad method). In what follows we'll use coordinate free formalism and modern apparatus of differential geometry. General scheme of the section is similar to [20], where one also can find some details in classical coordinate form.

2.1 Algebra of monads method

Let \mathcal{M} be (semi-)riemannian 5-dimensional manifold² with some fixed metric $G \equiv \langle , \rangle$. The most general way to go from the 5D world to some "embedded" 4D is to fix smooth *4D observers 1-form* λ (*monad field*):

$$\langle \lambda, \lambda \rangle = \varepsilon, \tag{3}$$

where we leave freedom of causal type of direction j_λ by means of constant factor $\varepsilon = \pm 1$. The form induces decompositions of tangent and cotangent spaces at every point $p \in \mathcal{M}$:

$$T_p\mathcal{M} = (T_p)_h\mathcal{M} \oplus j_\lambda(p); \quad T_p^*\mathcal{M} = (T_p^*)_h\mathcal{M} \oplus \lambda(p), \tag{4}$$

where *horizontal* tangent and cotangent spaces are:

$$(T_p)_h\mathcal{M} \equiv \{v \in T_p\mathcal{M} \mid \lambda(v)_p = 0\} \quad \text{and} \quad (T_p^*)_h\mathcal{M} \equiv \{\omega \in T_p^*\mathcal{M} \mid \omega(j_\lambda)_p = 0\}$$

respectively. The subspaces $\text{span}_p(j_\lambda)$ and $\text{span}_p(\lambda)$ we'll call *vertical*. Let's note, that the set

$$T_h\mathcal{M} \equiv \bigcup_{p \in \mathcal{M}} (T_p)_h\mathcal{M}$$

(or similarly $T_h^*\mathcal{M}$) in general does'nt admit local representation $R \times T(\mathcal{M}_h)$ where \mathcal{M}_h — *horizontal manifold*, since form λ can be *anholonomic (nonintegrable)*. In this situation we'll refer

²In this section our consideration will be local, so we don't fix global topology on \mathcal{M} .

to \mathcal{M}_h as *anholonomic horisontal manifold* [23], such that formally³ $T(\mathcal{M}_h) \equiv T_h\mathcal{M}$, keeping in mind that just the case is realized, when we observe 4D world filled by electromagnetic fields from the viewpoint of 5D KKT (see Sec.2.3 below).

Tensor continuations of (4) give decomposition of a whole tensor algebra $\mathcal{T}(\mathcal{M})$ on $\lambda - h$ components. Formally, let consider linear operator (affinor field):

$$\hat{h} \equiv \hat{I} - \varepsilon\lambda \otimes j_\lambda \equiv \hat{I} - \hat{\lambda}, \quad (5)$$

mapping $T\mathcal{M} \rightarrow T\mathcal{M}$ and $T^*\mathcal{M} \rightarrow T^*\mathcal{M}$. Here $\hat{I} = \text{id}_{T\mathcal{M}}$ or $\hat{I} = \text{id}_{T^*\mathcal{M}}$. By the definition it follows, that $\hat{h}(\hat{h}(X)) = \hat{h}(X)$ and $\langle \hat{h}(X), Y \rangle = 0$ for every vector field X and every vertical Y (the same is true for 1-forms). So, \hat{h} is projector: $T\mathcal{M} \xrightarrow{\hat{h}} T_h\mathcal{M}$ or $T^*\mathcal{M} \xrightarrow{\hat{h}} T_h^*\mathcal{M}$. Writing $\hat{I} = \hat{\lambda} + \hat{h}$ and taking its n -th tensor degree, we have:

$$\hat{I}^{\otimes n} \equiv \text{id}_{T_{n-r}^r(\mathcal{M})} = (\hat{\lambda} + \hat{h})^{\otimes n} = \sum_{\varsigma} \hat{\pi}_{\varsigma}, \quad (6)$$

where ς runs all binary sequences of symbols $\{\lambda, h\}$ of length n , $\hat{\pi}_{\varsigma}$ — projector on ς -th component of $T_{n-r}^r(\mathcal{M})$. Acting by initial and final operators of (6) on any tensor field $T \in T_{n-r}^r(\mathcal{M})$, we have

$$T = \sum_{\varsigma} T_{\varsigma}, \quad (7)$$

where $T_{\varsigma} = \hat{\pi}_{\varsigma}(T)$ — ς -th projection of T . In what follows we'll denote projections by index-like symbols λ or h when it will not lead to ambiguousness. For example, any vector field can be decomposed as follows: $X = \varepsilon X_{\lambda} j_{\lambda} + X_h$, where $X_{\lambda} \equiv \lambda(X)$, $X_h \equiv \hat{h}(X)$.

With using (7) it is easy to get decomposition of G :

$$G = \varepsilon\lambda \otimes \lambda + h, \quad (8)$$

where h is *metric on (anholonomic) manifold \mathcal{M}_h* , defined by the rule:

$$h(X, Y) = \langle \hat{h}(X), \hat{h}(Y) \rangle \quad (9)$$

for any vector fields X, Y . (9) means, that $h(X, Y) = G(X, Y)$ for every horizontal vector fields $X = X_h$ and $Y = Y_h$ and $\ker h = \text{span}(j_{\lambda})$.

Physically, any λ defines smooth family of 4D observers histories, which trace out 4D worlds inside the given 5D world. Accordingly to some modern concepts, suggested by brane physics and quantum mechanics, 1-form λ should be related to perceptive spaces of an observers [26], that we'll discuss in Conclusion.

2.2 1+4-analysis on \mathcal{M} .

By (3) it follows, that⁴ $(\nabla_{\lambda}\lambda)_{\lambda} = 0$. Let define *horizontal curvature 1-form of j_{λ} -congruence*:

$$\alpha \equiv \nabla_{\lambda}\lambda.$$

³In case of the, so called, *complete nonintegrability*, Rashevski-Chow's theorem [24, 25] states, that $\mathcal{M}_h = \mathcal{M}$ i.e. any two points of \mathcal{M} can be joined by a some horisontal curve γ_h .

⁴Here and below we use abbreviated notation $D_{\omega} \equiv D_{j_{\omega}}$ for any kind of derivative D along vector field j -conjugated with some 1-form ω .

It is obviously, that the tensor $\mathcal{H} \equiv \nabla\lambda - \varepsilon\lambda \otimes \alpha$ is horizontal. It can be decomposed on symmetric and antisymmetric components: $\mathcal{H} = \mathcal{D} + \mathcal{F}$, where

$$\mathcal{D} \equiv \hat{\mathcal{S}}(\nabla\lambda - \varepsilon\lambda \otimes \alpha) = \frac{1}{2}(\mathcal{L}_\lambda G - \varepsilon\lambda \vee \alpha)$$

— *4D extrinsic curvature tensor*,

$$\mathcal{F} \equiv \hat{\mathcal{A}}(\nabla\lambda - \varepsilon\lambda \otimes \alpha) = \frac{1}{2}(d\lambda - \varepsilon\lambda \wedge \alpha)$$

— *4D twist tensor* and $a \vee b \equiv a \otimes b + b \otimes a$. Finally, we obtain:

$$\nabla\lambda = \varepsilon\lambda \otimes \alpha + \mathcal{H}. \quad (10)$$

Acting in (10) by j from the right (with using $[\nabla, j] = 0$), we obtain for vector field j_λ :

$$\nabla j_\lambda = \varepsilon\lambda \otimes j_\alpha + \mathcal{H}j. \quad (11)$$

Following to [20], let define operators of *vertical* and *horizontal* (4D space-time) *derivatives*:

$$\dot{T}_h \equiv \frac{d}{d\lambda} T_h \equiv (\mathcal{L}_\lambda T_h)_h; \quad {}^{(4)}\nabla T_h \equiv (\nabla_h T_h)_h,$$

where T_h — arbitrary horizontal tensor field. On scalar functions by definition:

$$\dot{f} \equiv j_\lambda(f); \quad {}^{(4)}\nabla f \equiv (df)_h \equiv d_h f.$$

With using (7) the following identity for any vector field Z can be established:

$$\begin{aligned} \nabla Z = & {}^{(4)}\nabla Z_h + \varepsilon Z_\lambda \mathcal{H}j + (\dot{Z}_\lambda - Z_\alpha)\lambda \otimes j_\lambda + \lambda \otimes (Z_\lambda j_\alpha + \varepsilon(\dot{Z}_h + \mathcal{H}j(Z_h, \cdot))) \\ & + \varepsilon((dZ_\lambda)_h - \mathcal{H}(\cdot, Z_h)) \otimes j_\lambda, \end{aligned} \quad (12)$$

where $Z_\alpha = \alpha(Z)$. Acting on (12) by ι from the right, identifying $\iota_Z \equiv \omega$ and using the relation

$$\iota_{\dot{Z}_h} = \frac{d}{d\lambda} \iota_{Z_h} - 2\mathcal{D}(Z_h, \cdot),$$

we have for 1-forms:

$$\begin{aligned} \nabla\omega = & {}^{(4)}\nabla\omega_h + \varepsilon\omega_\lambda \mathcal{H} + (\dot{\omega}_\lambda - \omega_\alpha)\lambda \otimes \lambda + \lambda \otimes (\omega_\lambda \alpha + \varepsilon(\dot{\omega}_h - \mathcal{H}j(\cdot, \omega_h))) \\ & + \varepsilon((d\omega_\lambda)_h - \mathcal{H}j(\cdot, \omega_h)) \otimes \lambda. \end{aligned} \quad (13)$$

Assuming in (13) $\omega = \lambda$, $\omega_\lambda = \varepsilon$, $\omega_h = \omega_\alpha = 0$ we obtain (10).

The formulae (12)-(13) show, that any 5D expression, including covariant derivatives can be reexpressed in terms of vertical and horizontal derivatives. The following useful identities are easy checked:

$$\dot{\lambda} = \alpha = \mathcal{L}_\lambda \lambda; \quad {}^{(4)}\nabla \lambda \equiv (\nabla_h \lambda)_h = \mathcal{H}; \quad \dot{h} = 2\mathcal{D} = \dot{G}; \quad {}^{(4)}\nabla h = 0.$$

The latter expression suggests, that operator ${}^{(4)}\nabla$ should be treated as "covariant"⁵ (relatively h) derivative on \mathcal{M}_h .

⁵In fact, ${}^{(4)}\nabla$ possesses effective torsion, since direct calculation gives: $\text{Tors}_{(4)\nabla}(X_h, Y_h) \equiv {}^{(4)}\nabla_{X_h} Y_h - {}^{(4)}\nabla_{Y_h} X_h - [X_h, Y_h] = 2\varepsilon\mathcal{F}(X_h, Y_h)j_\lambda$. However, with respect to *horizontal bracket*: $[\cdot_h, \cdot_h]_h$ torsion of ${}^{(4)}\nabla$ is zero.

2.3 Effective matter lagrangian in 1 + 4-formalism

Twice substituting (12) into the definition of curvature operator:

$$\text{Riem}(X, Y)Z \equiv (\nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X, Y]})Z$$

and twice appropriately contracting the obtained expression, after some 1 + 4 algebra, outlined in previous subsection, we obtain: ${}^{(5)}R = {}^{(4)}R + M$, where ${}^{(4)}R$ — 4D scalar curvature and

$$M = 2\alpha^2 - 2\varepsilon \text{div } j_\alpha + 2\varepsilon \dot{\overline{\mathcal{D}}} + \varepsilon(\overline{\mathcal{D}}^2 + \mathcal{D}^2) + \varepsilon \mathcal{F}^2 \quad (14)$$

— matter scalar⁶. Here $T^2 \equiv \langle T, T \rangle$ for any tensor field T , $\text{div } X_h \equiv \text{Tr}({}^{(4)}\nabla X_h)$, $\overline{T} \equiv \text{Tr}(Tj)$ for any $T \in T_2^0(\mathcal{M})$.

So, 5D KKT inspires the following action for λ :

$$\mathcal{A}_m[\lambda] = -\frac{1}{2\kappa_5} \int_{\mathcal{M}} M \, d\text{vol}_5 \quad (15)$$

with M given by (14). (15) is starting point for our following consideration.

3 Nematic structure of 5D space-time

Since now we are interested by extreme form λ on \mathcal{M} with forgiven vacuum metric G , it would be more appropriately temporary to go aside from the standard view in KKT and rewrite (14) in terms of λ and its 5D covariant derivatives⁷. After little algebra (14) can be performed as follows:

$$M = -(\nabla_\lambda \lambda)^2 + 2\varepsilon j_\lambda (\text{Div } j_\lambda) + \varepsilon (\text{Div } j_\lambda)^2 - 2\varepsilon \text{Div } \nabla_\lambda j_\lambda + \varepsilon (\nabla \lambda)^2. \quad (16)$$

Before deriving equilibrium equations let's clear out nematic properties of space-time with lagrangian (16). For this purpose we need to remind general facts of common 3D nematic crystals physics and generalize it for 5D case. Nematic liquids form a subclass of fluid bodies with homogeneous but anisotropic correlation function, possessing axial symmetry [27]. In other words, at every point of nematic liquid there is direction, connected with orientation anisotropy of single molecules, which the liquid consist of. Macroscopically this situation can be described by means of unit vector field n , named *field of director*. It should be, in fact, understood as an element of unit projectivized tangent bundle $U_P \approx R^3 \times RP^2$, since directions n and $-n$ for nematic are physically equivalent. Absolute nematic energy minimum is realized under $n = \text{const}$, while nonuniform field n describes possible deformed state of nematic. Elastic (free) energy density of such deformed state within linear theory can be expressed through invariant quadratic combinations of derivatives ∇n possessing all required symmetry properties. Up to a boundary terms nematic elastic energy density has the following general kind [27]⁸:

$$F = \frac{K_1}{2} (\text{div } n)^2 + \frac{K_2}{2} (n, \text{rot } n)^2 + \frac{K_3}{2} (\nabla_n n)^2, \quad (17)$$

⁶The expression (14) corresponds to the formula (11.20) on p.218 in [20]. The correspondence is established with taking into account that: 1) author works in special (the, so called, chronometric) gauge of monad formalism, that requires fixing of coordinate system adopted to j_λ ; 2) curvature operator in [20] has opposite sign to accepted in present work; 3) $-\varepsilon \Phi$ in [20] corresponds to our α , $-\varepsilon \tilde{F}$ corresponds to \mathcal{F} . Our \mathcal{D} is the same as in [20].

⁷Direct calculations with (14) are also possible, but take much more efforts, since 4D and λ -derivatives don't commute with variations δ , while for example $[\nabla, \delta] = 0$ since G is fixed.

⁸ There is misprint in this book in expression for invariant $(\nabla n)^2$ in §140. Right (and exact) expression is $(\nabla n)^2 = (n, \text{rot } n)^2 + (\text{div } n)^2 + (\nabla_n n)^2 + \text{div}(\nabla_n n - n \text{div } n)$.

where K_1, K_2, K_3 — *Frank's moduluses*, responsible for *splay*, *twist* and *bend* nematic elasticity respectively (here temporary div means 3D divergency, rot — standard 3D curl, ∇ — covariant derivative in 3D euclidian space, $(\ , \)$ — 3D euclidian scalar product). Some interesting problems, concerning static nematic deformations and their topological properties, whose 5D analogies we'll consider lately, can be found in [28] (§36-39).

Easy to see, that 5D space-time in KKT can be treated as some nematic medium in the problem of extreme λ finding, since the lagrangian $L_\lambda = -(1/2\kappa_5)M$ with M given by (16) has similar to (17) structure. This remarkable analogy suggests once again that space-time (4D or multidimensional) can manifest properties of continuum media in various aspects, — the fact, that make such analogies useful for studying, interpreting and modeling of space-time physics.

To express (16) in terms of Frank's moduluses we need express it in terms of independent quadratic invariant combinations, which are 5D generalizations of those in (17). The first and second terms in (17) have trivial generalizations:

$$(\text{div } n)^2 \rightarrow (\text{Div } j_\lambda)^2; \quad (\nabla_n n)^2 \rightarrow (\nabla_\lambda \lambda)^2.$$

The expression $(n, \text{rot } n)^2$, which is called *anholonomicity of 3D vector field n* , has direct generalization $(\lambda \wedge d\lambda)^2/3!$, since both expressions guarantee local integrability of 1-forms ι_n and λ respectively and in 3D euclidian space second is identical to the first. Using relations:

$$X(f) \stackrel{b}{=} -f \text{Div } X; \quad (\lambda \wedge d\lambda)^2 \stackrel{b}{=} 3!(\varepsilon((\nabla \lambda)^2 - (\text{Div } j_\lambda)^2) - (\nabla_\lambda \lambda)^2)$$

where X, f — arbitrary vector field and scalar function respectively, " $\stackrel{b}{=}$ " means "is equal up to a total divergence" (the second identity in 3D space under $\varepsilon = 1$ is the formula of footnote 8 up to a boundary terms), we obtain from (16):

$$\mathcal{L}_\lambda \stackrel{b}{=} -\frac{1}{12\kappa_5}(\lambda \wedge d\lambda)^2. \tag{18}$$

From (18) we see that:

1. *Nematic elasticity of 5D space-time, inspired by 5D KKT, concerns only nematic twists;*
2. *Nonzero Frank's modulus $K_2 = -1/6\kappa_5$ is induced by 5D gravity.* The similar relation between Young's modulus of multidimensional space-time and 4D Einstein constant \varkappa has been observed in [9] in the context of "common" elasticity of space-time;
3. *From the view point of 4D physics 5D nematic elasticity characterizes resistance of 5D space-time with respect to anholonomicity of embedded 4D physical worlds.* In other words, 5D "nematic vacuum" contains only holonomic physical world(s), traced out by 1-form λ with $\lambda \wedge d\lambda = 0$.

4 Variational problem

Varying modified action (15) with lagrangian (16)⁹:

$$\mathcal{A}[\lambda] = -\frac{1}{2\kappa_5} \int_{\mathcal{M}} (M - Q[\lambda^2 - \varepsilon]) \, d\text{vol}_5,$$

⁹We go back from lagrangian (18) to (16) in order to obtain right boundary conditions.

including Lagrange multiplier Q , after standard extracting of exact forms we obtain the following volume part of variation:

$$\delta \mathcal{A}_{\text{vol}} = -\frac{1}{\kappa_5} \int_{\mathcal{M}} \delta \lambda \left[\nabla_{\lambda}^2 j_{\lambda} - \langle \text{Grad } \lambda, \nabla_{\lambda} \lambda \rangle + \text{Div } j_{\lambda} \nabla_{\lambda} j_{\lambda} + \varepsilon (\text{Grad Div } j_{\lambda} - \nabla^2 j_{\lambda}) - Q j_{\lambda} \right] d\text{vol}_5 \quad (19)$$

and boundary terms

$$\delta \mathcal{A}_{\text{b}} = -\frac{1}{\kappa_5} \int_{\partial \mathcal{M}} \left[(\varepsilon \text{Div } \delta j_{\lambda} - \langle \delta \lambda, \nabla_{\lambda} \lambda \rangle) \pi_{j_{\lambda}} - \varepsilon (\pi_{\nabla_{\lambda} j_{\lambda}} + \pi_{\nabla_{\lambda} \delta j_{\lambda}}) + \varepsilon \pi_{j_{(\nabla \lambda)(\cdot, \delta j_{\lambda})}} \right] d\text{vol}_5,$$

By arbitrariness of $\delta \lambda$ (19) gives the following 5D covariant nematic equilibrium equations:

$$\nabla_{\lambda}^2 j_{\lambda} - \langle \text{Grad } \lambda, \nabla_{\lambda} \lambda \rangle + \text{Div } j_{\lambda} \nabla_{\lambda} j_{\lambda} + \varepsilon (\text{Grad Div } j_{\lambda} - \nabla^2 j_{\lambda}) - Q j_{\lambda} = 0 \quad (20)$$

Its λ -component defines Lagrange multiplier Q :

$$\varepsilon Q = (\nabla_{\lambda}^2 j_{\lambda})_{\lambda} - (\nabla_{\lambda} \lambda)^2 + \varepsilon (\nabla_{\lambda} \text{Div } j_{\lambda} - (\nabla^2 j_{\lambda})_{\lambda}).$$

Physical meaning has h -component of (20):

$$(\nabla_{\lambda}^2 j_{\lambda})_h - \langle \text{Grad}_h \lambda, \nabla_{\lambda} \lambda \rangle + \text{Div } j_{\lambda} \nabla_{\lambda} j_{\lambda} + \varepsilon (\text{Grad}_h \text{Div } j_{\lambda} - (\nabla^2 j_{\lambda})_h) = 0. \quad (21)$$

5 1+4-form of nematic equations

For interpreting of equations (21) it is more convenient to go again to 1 + 4-representation. Using the identities:

$$(\nabla_{\lambda}^2 j_{\lambda})_h = j_{\alpha} + \mathcal{H} j(j_{\alpha}, \cdot); \quad \langle \text{Grad}_h \lambda, \nabla_{\lambda} \lambda \rangle = j \mathcal{H}(\cdot, j_{\alpha}); \quad \text{Div } j_{\lambda} \nabla_{\lambda} j_{\lambda} = \overline{\mathcal{D}} j_{\alpha};$$

$$\text{Grad}_h \text{Div } j_{\lambda} = d_h \overline{\mathcal{D}}; \quad (\nabla^2 j_{\lambda})_h = \varepsilon \overline{\mathcal{D}} j_{\alpha} + \varepsilon j_{\alpha} + \varepsilon \mathcal{H} j(j_{\alpha}, \cdot) + \text{div } \mathcal{H},$$

equations (21) can be rewritten in the following equivalent 1 + 4 form:

$$\varepsilon \text{div } \mathcal{F} j = \varepsilon \text{grad } \overline{\mathcal{D}} - \varepsilon \text{div } \mathcal{D} j - j \mathcal{H}(\cdot, j_{\alpha}), \quad (22)$$

where $\text{div } T_h = \overline{({}^4\nabla} j T_h$, $\text{grad} \equiv j({}^4\nabla)$. In such form, it can be interpreted as follows: *origins of twists of \mathcal{M}_h (and consequently anholonomicity, since $\lambda \wedge d\lambda = 2\lambda \wedge \mathcal{F}$) are nonhomogeneous deformations and curvature of congruence j_{λ} .* Another (equivalent) interpreting of nematic equilibrium equations can be by means of Kaluza-Klein-Maxwell equations:

$$\text{Ric}_{\lambda h} = 0 \quad \Leftrightarrow \quad \varepsilon \text{div } \mathcal{F} j = -\varepsilon \text{grad } \overline{\mathcal{D}} + \varepsilon \text{div } \mathcal{D} j + 2\mathcal{F} j(j_{\alpha}, \cdot).$$

Their combination gives the following, equivalent system of nematic equilibrium equations:

$$2\varepsilon \text{div } \mathcal{F} j = (3\mathcal{F} - \mathcal{D}) j(j_{\alpha}, \cdot). \quad (23)$$

which doesn't contain derivatives of \mathcal{D} .

6 Particular solutions

Let's consider some particular solutions to (23), which corresponds to some earlier used λ .

1. $\mathcal{F} = \mathcal{D} = 0$. This choice has been used by a number of authors [20, 21], who have investigated 4D physical properties of a "fifth coordinate independent" 5D physical world without electromagnetism. An effective matter of the models is originated only from j_α , which in special coordinate system, adopted to j_λ , is proportional to $\text{grad } \varphi$, where $\varphi = \sqrt{|G_{55}|}$ — geometrical scalar field.

2. $\mathcal{F} = 0$ $j_\alpha = 0$. This choice has been used in works by a number of other authors [16], where an effective matter involves "fifth coordinate dependency" of 5D metric. Easy to see, that, the canonical frame of 5D metric $G = g_{\alpha\beta}(x, \eta)(dx^\alpha \otimes dx^\beta) + \varepsilon d\eta \otimes d\eta$, $\alpha, \beta = 0, 1, 2, 3$, introduced in [16] and in a number of earlier works of the author, just can be related to the considered particular class of solutions to (23).

3. $\mathcal{F} = 0$; $\mathcal{D}(j_\alpha, \cdot) = 0$. This class is intermediate between 1 and 2. It has'nt been investigated in literature.

All considered cases imply $\mathcal{F} = 0$, in spite of the central idea of KKT — geometrization of electromagnetic interactions. We'll discuss this circumstance in Conclusion.

7 Example: nematic structure of a flat 5D space-time

Let \mathcal{M} be flat 5D Minkowski space-time with metric G :

$$G = dt \otimes dt - dr \otimes dr - r^2 d\varphi \otimes d\varphi - dz \otimes dz - d\eta \otimes d\eta,$$

taken in 5D cylindrical coordinate system. We'll treat \mathcal{M} as infinite nematic medium with no boundaries. Lets consider the following situations.

1. $\lambda = rd\varphi$. Direct calculations gives:

$$\alpha = d \ln r; \quad \mathcal{F} = \mathcal{D} = 0,$$

so equilibrium equations are satisfied identically (case 1). 4D space-times $\mathcal{M}_h(\varphi)$, defined by λ , are 4D pseudoeuclidian hyperplanes, all going through 3-plane $P_3 : r = 0$ (see Fig.1(1)).

2. $\lambda = dr$. Direct calculations give:

$$\alpha = 0; \quad \mathcal{F} = 0; \quad \mathcal{D} = rd\varphi \otimes d\varphi,$$

so equilibrium equations are satisfied identically (case 2). Here physical worlds $\mathcal{M}_h(r)$, defined by λ , are 4D pseudoeuclidian coaxial cylinders (see Fig.1(2)). 3-plane $r = 0$ is peculiar only in the sense that: $\dim \mathcal{M}_h(0) = 3$, rather than 4.

3. $\lambda = (1/\sqrt{\Delta})((r^2/r_0)d\varphi + dz)$, where $\Delta = 1 + (r/r_0)^2$, $r_0 = \text{const}$. Direct calculations give:

$$\alpha = \frac{r}{r_0^2 \Delta} dr; \quad \mathcal{F} = \frac{r}{r_0 \Delta^{3/2}} (dr \wedge d\varphi - \frac{1}{r_0} dr \wedge dz); \quad \mathcal{D} = 0.$$

This solution possesses nonzero nematic energy density and corresponds to the solution¹⁰ of problem 1 in §38 of [28]. Total 5D nematic energy of configuration (per element of infinite 3D volume), is

$$\mathcal{E} = \frac{K_2}{12} \int_{R^2} r dr d\varphi (\lambda \wedge d\lambda)^2 = -2\pi K_2.$$

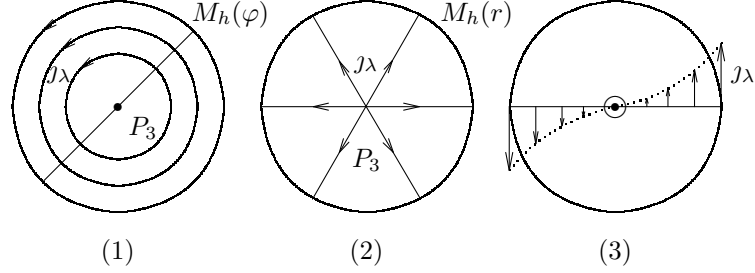


Figure 1: Nematic structures with cylindrical symmetry. In case (3) horizontal projection of field j_λ is shown. The vector field turns about radial direction and become vertical (orthogonal to a picture plane) at the center.

Under $r \rightarrow \infty$ $\lambda_\infty = rd\varphi$, under $r \rightarrow 0$ $\lambda_0 = dz$. In Fig.1(3) nematic structure of the solution is performed.

Examples 1 and 2 perform nematic structures with linear topological defects, which are called *disclinations*. We'll discuss them in Conclusion.

8 Conclusion

1. Nature of 5D nematic structure. Nematic structure of common fluid crystals is originated from special kind of interaction between individual molecular dipoles. Generic thread-like nematic structure provides total dipole-dipole interaction energy minimum by means of more or less clear 3D physical mechanism.

In case of 5D KKT there is no such clear physical mechanism of how individual vectors $j_\lambda(p)$ form smooth vector field j_λ on \mathcal{M} . Moreover, we don't know even what is physical nature of an individual vector $j_\lambda(p)$, attached to every point $p \in \mathcal{M}$. However, the role of j_λ in KKT — extracting of 4D *observable* physics from 5D geometry, suggests, that nature of j_λ should concern some subtle aspects of relations between observer consciousness and multidimensional physical (or more exactly geometrical) reality. This view adjoins Penrose's "physics of consciousness" (PhC) [29] in the multidimensional physics context. Present work should be regarded as "macroscopic" (or "phenomenological") model of PhC, which don't touch origins and internal structure¹¹ of λ . But even this "averaged" approach, inspired by 5D KKT, suggests the following conclusion: *it would be no 4D space-time and matter without nematic structure of 5D space-time*. When 5D nematic structure is destroyed (due to some extreme conditions, originated from a more rich 5D geometrical model, including nonlinearity and thermodynamics), 4D space-time and matter disappear and we have chaotically distributed horizontal tangent spaces in \mathcal{M} or "space-time-matter chaos."

2. Boundary conditions. In common nematic crystals physics boundary interactions of nematic molecules as a rule are much more intensive then volume ones. It allows operate nematic structure in laboratory experiments. From the theoretical viewpoint boundary conditions fix integration constants in general solutions to equilibrium equations.

In case of 5D nematic, viewed as object of PhC, status of 5D boundary conditions is unclear. Probably, we should reverse our consideration: fixing integration constants in general solution by comparing 4D physics, involved by λ , with observable 4D world, one could try to determine (may be under some additional assumptions) 5D boundary conditions (see discussion in [30]).

¹⁰The solution, obtained in this book is valid under $K_3 > K_2$. Our solution ($K_3 = 0$) is not contained in those, performed on p.203 of [28].

¹¹Similarly to statistical physics of liquid crystals, λ could be viewed as some collective property of "mental molecules." In this context monad field λ resembles Leibniz's metaphysical "monad" not only terminologically.

3. Interaction with curvature and others fields. In present work we have restricted ourself by the simplest 5D model — pure geometry of vacuum 5D space-time. It has led us to a rather "poor" physics, involving only twist deformations of 5D nematic media. Contrary to this 5D model, common 3D nematics, possessing all Frank's moduluses, are able to form rich topological structures due to a subtle balance of splay, twist and bend energies. Moreover, external electromagnetic fields can operate director's field inside nematic sample. This property is widely used in fluid crystal screens (Freedericksz's effect).

The similar situation can be obtained in 5D nematic by suitable generalization of our model. 5D Ricci curvature, nonlinearity or nonriemannian objects – torsion and nonmetricity – will force 5D nematic structure as some "external fields." We put off all this possibility for a future work.

4. Physics of nematic topological defects. Let's turn our attention to example 1 of Sec.7 with circular nematic structure. In spite of flatness 4D worlds $\mathcal{M}_h(\varphi)$ possess the remarkable property: *since $\lambda|_{P_3} = 0$, then $\ker \lambda|_{P_3} = (T\mathcal{M})|_{P_3}$, i.e. tangent to peculiar 3D pseudoeuclidian plane spaces are 5D, rather than 4D (or 3D).* What are physical manifestation of such peculiarity?

Let some test particle moves along 5D geodesic $\gamma(\tau)$ with 5D velocity U , satisfying 5D geodesic equations: $\nabla_U U = 0$. When such geodesic will be horizontal (i.e. $d\gamma/d\tau \in T_h\mathcal{M}$ for every $\tau \in R$)? In case $U_\lambda = \dot{U}_\lambda = 0$ at some point of γ , decomposition (12) gives at the same point:

$$\nabla_U U = {}^{(4)}\nabla_{U_h} U_h - \varepsilon \mathcal{D}(U_h, U_h) j_\lambda$$

or equivalently

$$\nabla_U U = 0|_{U_\lambda=0} \Leftrightarrow {}^{(4)}\nabla_{U_h} U_h = 0; \quad \mathcal{D}(U_h, U_h) = 0.$$

We see, that free test particle, moving at some point with horizontal velocity U_h , will continue to move along 4D geodesic, if $\mathcal{D}(U_h, U_h) = 0$ at any $\tau \in R$. In our example $\mathcal{D} = 0$, so any free test particle, living on some regular part of $\mathcal{M}_h(\varphi)$, will move there rectilinearly, always being attached to the $\mathcal{M}_h(\varphi)$.

This picture is violated on P_3 . Here horizontal tangent space, spanned by all possible directions of initial velocities, is whole $T\mathcal{M}|_{P_3}$ (or, more exactly, interior of 5D light cone). It means, that P_3 is 3D region of 4D worlds, where one can send particles and signals in extradimension or receive them from there. In other words, P_3 could be looked as the place, where 4D conservation laws are violated (while 5D ones are, of course, valid).

Another peculiar property of P_3 follows from the relation:

$$P_3 = \bigcap_{\varphi=0}^{2\pi} \mathcal{M}_h(\varphi).$$

So, P_3 also can be viewed as "junction station" for travels¹² from one $\mathcal{M}_h(\varphi_1)$ to another $\mathcal{M}_h(\varphi_2)$.

5. Topological classification of defects and Frank's indexes. Circular and radial defects, shown in Fig.1, are particular cases of nematic disclinations, possessing cylindrical symmetry. Simple physical considerations (radial self-similarity and uniqueness, see [28, §39]) show, that any cylindrical disclination can be described by a winding number $n = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$, which is equal to a number of director revolutions under moving along closed path, embracing defect line. In case of cylindrical symmetry n is called *Frank's index* of topological defect. Vicinity of a defect with Frank's index n possesses axis of symmetry D_m , where $m = 2|n - 1|$, which, besides rotations by angles $\varphi = 2\pi p/m, p = 0, \dots, m$, includes reflections with respect to horizontal plane, orthogonal to axis of the defect. Note, that the defects, considered in examples 1 and 2 both have

¹²Note, that traveling objects should be matter points, lines or planes, since P_3 is 3D pseudoeuclidian space with 2D space section.

Frank's index $n = 1$. More deep topological analysis reveals, that all defects with integer n can be eliminated by continuous deformation of director field, while disclinations with half-integer n are topologically inherent. Mathematically this facts are originated from a structure of fundamental group of nematic configuration space RP^2 . As it can be shown by algebraic topology methods [31], the fundamental group $\pi_1(RP^2) = \mathbb{Z}_2$. In other words, all closed contours on RP^2 are divided on the two subclasses: closed and "semiclosed", with end points lying on diameter of projective sphere RP^2 . Integer Frank's indexes correspond to the first class, half-integer – to the second. Physially, inherent disclinations are topologically stable and can be observed in laboratory, while eliminable ones are destroyed by small external uncontrolled influences. Point-like defects of 3D nematic can be also topologically classified by means of structure of second fundamental group $\pi_2(RP^2) = \mathbb{Z}$. It turns out, that point-like defect is stable if its topological number (named sometimes "topological charge") $n \neq 0$.

In spite of more possibilities of n -dimensional nematic structures, topologically they copy 3D case. Really, configuration space of n -dimensional nematic is RP^{n-1} . Topological classification of k -dimensional defects ($0 \leq k \leq n-2$) will be established on structure of fundamental group $\pi_{n-k-1}(RP^{n-1})$. But all this group are well known:

$$\pi_{n-k-1}(RP^{n-1}) = \begin{cases} \mathbb{Z}, & k = 0; \\ 0, & 0 < k < n-2; \\ \mathbb{Z}_2, & k = n-2. \end{cases}$$

So, the case $k = 0$ is topological analog of point-like defects of 3D nematic, while multidimensional topological analog of line defect in 3D is the case $k = n-2$. All defects of intermediate dimensions are topologically trivial.

In a difference with topology, group theoretical structure of symmetry of multidimensional defect vicinity will be really more rich, then in case of 3D. The similar to 3D case considerations (in flat n -dimensional Minkowski space) lead to conclusion, that director field in vicinity of defect (of any dimension $0 \leq k \leq n-2$) will have symmetry of some discrete subgroup $\mathcal{S} \subset O(1, n-1)$, whose elements (their number and type) can be viewed as "generalized Frank's indexes" of the multidimensional nematic defects.

8. Physics on anholonomic manifold. Due to the fact, that KKT consider twist tensor \mathcal{F} both geometrically – as object, responsible for anholonomicity of horizontal 4D space-time, and physically – as (related to) geometrized electromagnetic strength tensor field, *Kaluza-Klein electrodynamics is not identical to standard Maxwell one*. This fact had led authors, who had worked with KKT, to a number of "fine tunings" of the theory, which had made equations of Kaluza-Klein electrodynamics compatible with Maxwell equations. For example, general \mathcal{F} is not suitable candidate for direct geometrization of electromagnetic field, since $d\mathcal{F} \neq 0$. One of the ways to obtain standard "second pair" of Maxwell equations is to specialize some geometrical objects. Let α be exact form, i.e. $\alpha = d \ln \psi$ and let¹³ $\varepsilon = +1$. Then, assuming electromagnetic potential form $A = (1/2)\sqrt{\kappa_5/8\pi l_5}\psi\lambda$, where l_5 — "size of \mathcal{M} in fifth dimension", we obtain:

$$dA = (1/2)\sqrt{\kappa_5/8\pi l_5}(d\psi \wedge \lambda + \psi d\lambda) = \sqrt{\kappa_5/8\pi l_5}\psi\mathcal{F}.$$

Then after 4D conformal transformation $h = \psi^{-1}\tilde{h}$, where \tilde{h} — physical (observable) 4D metric, we formally obtain from (15) "right geometrized action" for electromagnetic field:

$${}^{(5)}\mathcal{A}_{\text{em}} = -\frac{1}{16\pi} \int_{\mathcal{M}} (dA)^2 d\text{vol}_5. \quad (24)$$

¹³Opposite sign $\varepsilon = -1$, accepted in [20] is originated from the opposite sign on definition of curvature.

But if $\lambda \wedge d\lambda = 2\lambda \wedge \mathcal{F} \neq 0$, i.e. (after projection on \mathcal{J}_λ) $\mathcal{F} \neq 0$, then

$$\mathcal{M} \neq \bigcup_{\eta \in R} \mathcal{M}_h(\eta),$$

where $\mathcal{M}_h(\eta)$ — classical submanifolds of \mathcal{M} , i.e. *separate integration over extradimension in (24) is impossible*. Formal decomposition $d\text{vol}_5 = \lambda \wedge d\text{vol}_4^h$ and integration over \mathcal{M}_h with volume form $d\text{vol}_4^h$ will give infinite integrals, since by Rashewski-Chow's theorem anholonomic manifold \mathcal{M}_h is 5D (as a set) and it will have infinite measure with respect to 4D volume form $d\text{vol}_4^h$. Roughly speaking, *within 5D KKT we are able to derive 4D differential (local) laws of physics, induced by 5D geometry, while integral laws, generally speaking, are absent (or should be modified)*. So, standard Gauss theorem and Coulomb's law will be different from those, inspired by 5D KKT, since sphere, which is commonly used in derivation of Coulomb's law from Maxwell equations, on anholonomic manifold has nothing to do with common sphere (see [23]).

References

- [1] M.Born, *Phys.Zshr.* **12**, 569-575 (1911).
- [2] A.D.Sacharov, *DAN SSSR* **177**, 70, (1967) (In Russian).
- [3] V. F. Hehl, I. Newman, In pr-s "*Perspectivy edinoi teorii*", MSU, 137-166, 1991 (In Russian); also F. Gronwald and F. Hehl, gr-qc/9701054.
- [4] A. Tartaglia, *Grav. & Cosm.* **1**, 335 (1995) (gr-qc/9509043).
- [5] G. E. Volovik, *J.Low.Temp.Phys.* **24**, 127-129 (1998) (cond-mat/9706172).
- [6] A.Unzicker, gr-qc/0011064.
- [7] E. B. Eltsov and others, *Phys.Rev.Lett.* **85**, 4739-4742 (2000) (cond-mat/0007369).
- [8] M.Visser, C.Barcelo, S.Liberati, *Gen.Rel.Grav.* **34**, 1719-1734 (2002) (gr-qc/0111111).
- [9] S. S. Kokarev, *Nuovo Cimento B* **113**, 1339 (1998) (gr-qc/0010005).
- [10] S. S. Kokarev, *Nuovo Cimento B* **114**, 903 (1999) (gr-qc/0010038).
- [11] S. S. Kokarev, *Nuovo Cimento B* **116**, 915 (2001) (gr-qc/0108007).
- [12] S. S. Kokarev, *Grav. & Cosm.* **8** (Suppl.II), 75-78 (2002) (hep-th/0201014).
- [13] T.Kaluza, *Sitzungsber.d.Berl.Acad.* 966-971 (1921).
- [14] O.Klein, *Z.F.Phys.* **37**, 896-906 (1926).
- [15] Yu.B.Rumer, *Investigations on 5-optics*, Moscow, GITTL, 1956 (In Russian).
- [16] P.Wesson, *Journ. of Math. Phys.* **43**, 2423-2438 (2002) (gr-qc/0105059).
- [17] J.Ponce de Leon, *Int.J.Mod.Phys.* **D11**, 1355-1380 (2002) (gr-qc/0105120).
- [18] Yu.S.Vladimirov, S.S.Kokarev, *GRG* **29**, 141-151 (1997) (gr-qc/0210067).
- [19] G.Horowitz, K.Maeda, *Class.Quant.Grav.* **19**, 5543-5556 (2002) (hep-th/0207270).
- [20] Yu. S. Vladimirov, *Reference frames in gravitation theory*, M.Energoizdat, 1982 (In Russian).

- [21] S. S. Kokarev, *Grav. & Cosm.* **4**, 238-249 (1998) (gr-qc/0212098).
- [22] B. O'Neill, *Semi-riemannian geometry*, Acad. Press, Diego, California, 1983.
- [23] F.Griffits, *External differential systems and calculus of variations*, IO MFMI, 1999 (tr. from English), see addition by A.M.Vershik, Yu.A.Gershkovich "Anholonomic problems and geometry of distributions" there.
- [24] P.K.Rashevski, *Uch. zapiski Mosk. Ped. Inst. im. Libknehta*, (seriya fiz-mat.) N2, 83-94 (1938) (In Russian).
- [25] W.L.Chow, *Math. Ann.* **117**, N1 98-105 (1940).
- [26] S. S. Kokarev, *In Abstr. of V-th Asian-Pacific Conf.* 37-38, PFUR, Moscow, 2001.
- [27] L. D. Landau, E. M. Lifshits, *Statistical physics (v.5)* M. Nauka, 1995 (In Russian).
- [28] L. D. Landau, E. M. Lifshits, *Elasticity theory (v.7)* M. Nauka, 1987 (In Russian).
- [29] R.P.Penrose, *The Emperor's New Mind*, Oxford Univ. Press, 1989.
- [30] S.S.Kokarev, *Grav. & Cosm.* **7**, 67-73 (2001).
- [31] Yu.G.Borisovich and others, *Introduction to topology*, M, Nauka, Fizmatlit, 1995.